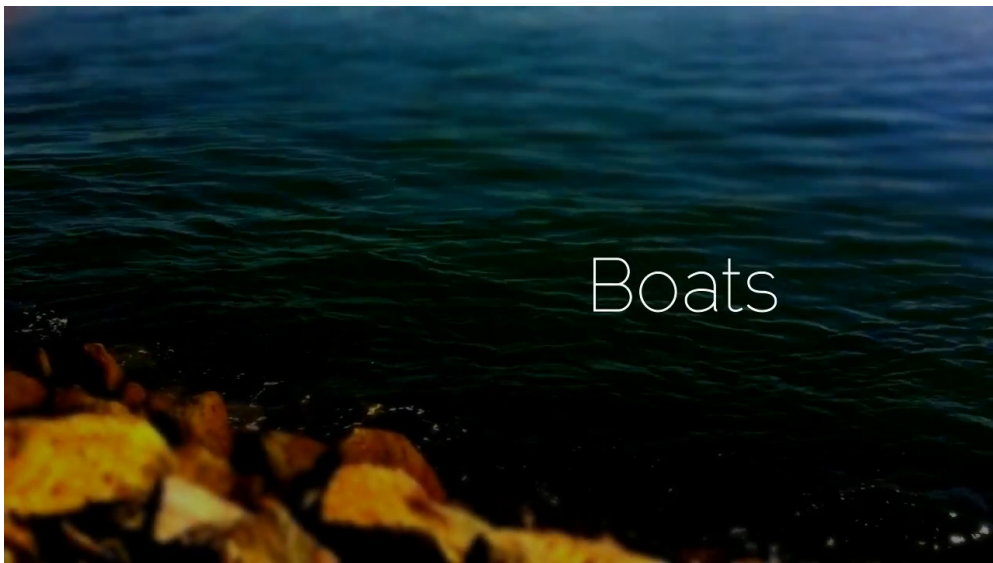


Course Notes

Optimization Problem: Minimum Distance Between Two Boats

Qwen3.6

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Video Author/Channel: integralCALC
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1 Problem Statement

The video presents a classic calculus optimization problem involving related rates and distance minimization. The scenario involves two boats moving in perpendicular directions relative to a central dock.



Figure 1: Problem statement displayed on screen. ¹

1.1 Scenario Details

- **Boat 1:** Leaves the dock at 2:00 p.m. and travels **South** at a constant speed of 20 km/h.
- **Boat 2:** Has been heading **East** at a constant speed of 15 km/h and is scheduled to reach the dock at 3:00 p.m.

1.2 Objective

The goal is to determine the exact time at which the two boats are closest to each other.

2 Mathematical Modeling

To solve this problem, we must translate the physical movement of the boats into mathematical functions. This requires setting up a coordinate system and defining the position of each boat as a function of time.

2.1 Coordinate System Setup

We place the dock at the origin $(0, 0)$ of a Cartesian coordinate system.

- The positive y -axis points North, and the negative y -axis points South.
- The positive x -axis points East, and the negative x -axis points West.

¹Video frame time interval: 00:13–00:27.

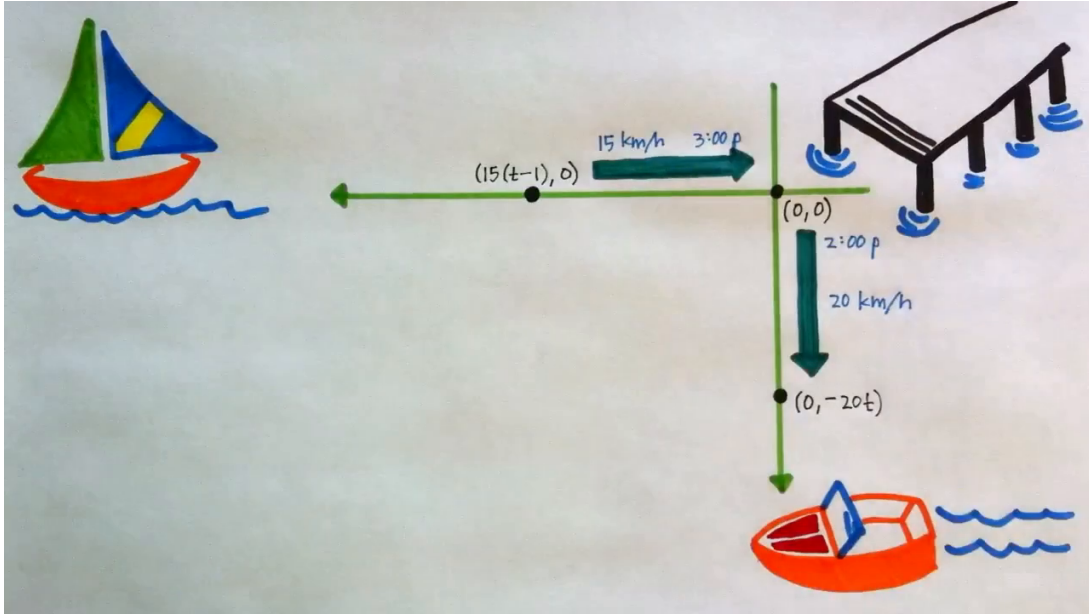


Figure 2: Diagram showing the coordinate system, boat positions, and velocity vectors. ²

2.2 Defining the Time Variable

Let t represent the time in hours elapsed since 2:00 p.m.

- At 2:00 p.m., $t = 0$.
- At 3:00 p.m., $t = 1$.

Time Definition Caution

Ensure that $t = 0$ corresponds to the start time of the first boat (2:00 p.m.). The second boat reaches the dock at 3:00 p.m., which corresponds to $t = 1$. This offset is crucial for correctly modeling the second boat's position.

2.3 Position Functions

Boat 1 (Southbound):

- Starts at $(0,0)$ at $t = 0$.
- Moves in the negative y direction at 20 km/h.
- Position at time t : $P_1(t) = (0, -20t)$.

Boat 2 (Eastbound):

- Moves in the positive x direction at 15 km/h.
- Reaches the origin $(0,0)$ at $t = 1$.
- Since it is moving East towards the origin from the West, its x -coordinate is negative for $t < 1$.
- The position function is $x(t) = 15(t - 1)$.
 - Check: At $t = 1$, $x = 0$ (at dock).
 - Check: At $t = 0$, $x = -15$ (15 km West of dock).
- Position at time t : $P_2(t) = (15(t - 1), 0)$.

²Video frame time interval: 00:47–01:12.

3 Optimization Process

We need to minimize the distance between $P_1(t)$ and $P_2(t)$.

3.1 Distance Function

The distance d between the two boats at time t is given by the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

To simplify the calculation, we minimize the square of the distance, d^2 . The value of t that minimizes d^2 will also minimize d , since distance is non-negative.

$$\begin{aligned} d^2 &= [15(t-1) - 0]^2 + [0 - (-20t)]^2 \\ d^2 &= (15t - 15)^2 + (20t)^2 \end{aligned}$$

3.2 Expanding the Function

Expanding the terms:

$$\begin{aligned} d^2 &= (225t^2 - 450t + 225) + 400t^2 \\ d^2 &= 625t^2 - 450t + 225 \end{aligned}$$

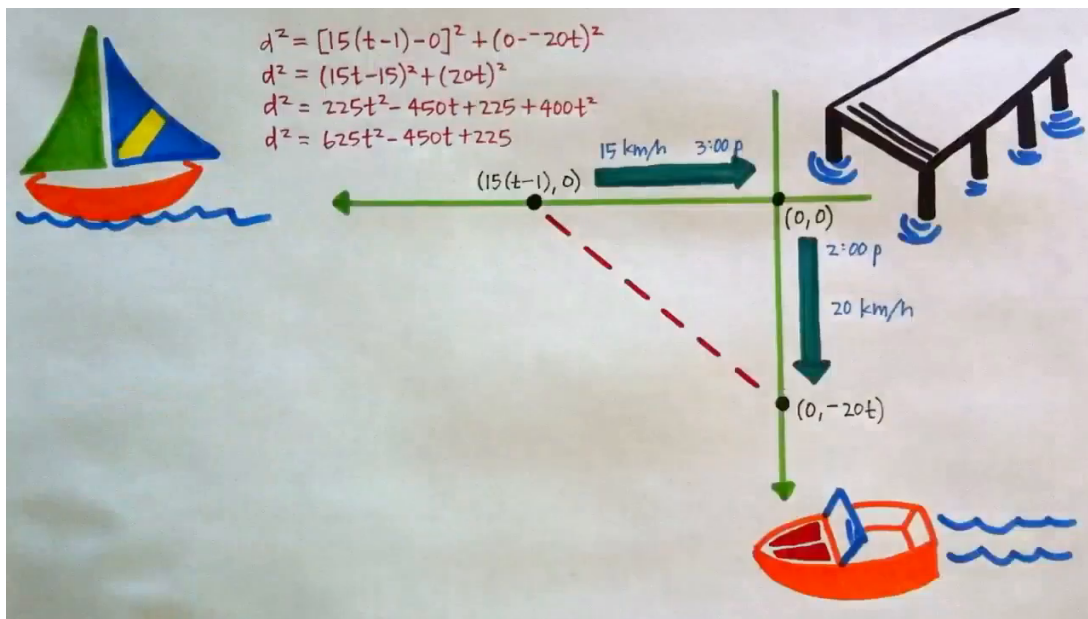


Figure 3: Derivation of the distance squared function. ³

3.3 Finding the Critical Point

To find the minimum distance, we take the derivative of d^2 with respect to t and set it to zero.

$$\begin{aligned} \frac{d}{dt}[d^2] &= \frac{d}{dt}(625t^2 - 450t + 225) \\ \frac{d}{dt}[d^2] &= 1250t - 450 \end{aligned}$$

Setting the derivative to zero:

$$\begin{aligned} 1250t - 450 &= 0 \\ 1250t &= 450 \end{aligned}$$

³Video frame time interval: 01:50-01:54.

$$t = \frac{450}{1250} = \frac{45}{125} = \frac{9}{25} \text{ hours}$$

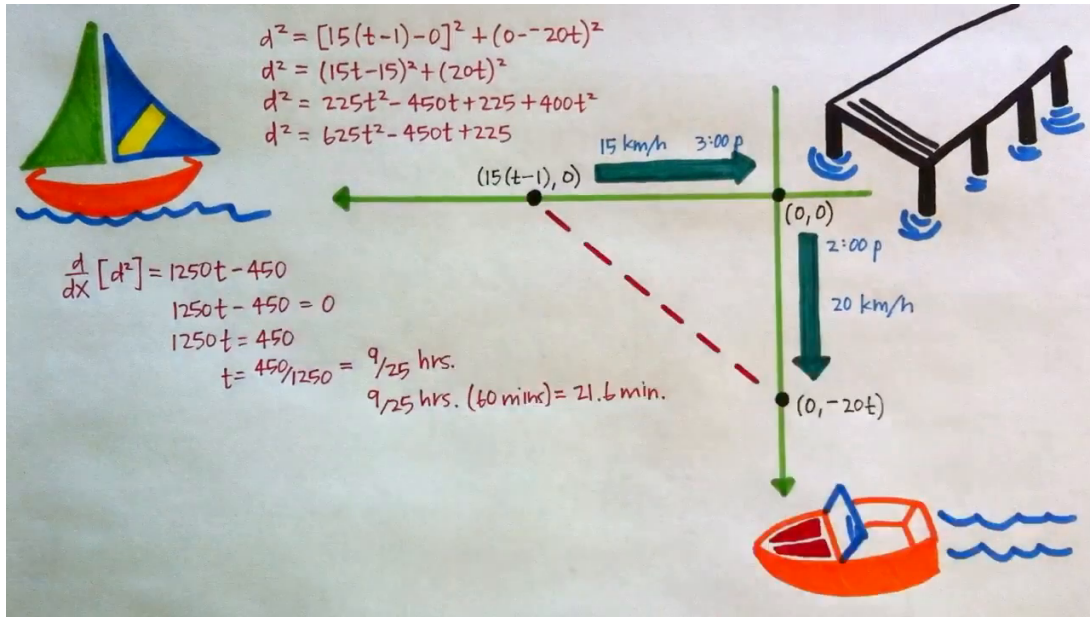


Figure 4: Solving for t using the derivative. ⁴

4 Solution and Interpretation

4.1 Calculating the Time

The boats are closest when $t = \frac{9}{25}$ hours. We convert this to minutes to find the clock time.

$$\text{Minutes} = \frac{9}{25} \times 60 = \frac{540}{25} = 21.6 \text{ minutes}$$

Since 0.6 minutes is $0.6 \times 60 = 36$ seconds, the time elapsed is 21 minutes and 36 seconds. Adding this to the start time of 2:00 p.m.:

Time of closest approach = 2:21:36 p.m.

4.2 Calculating the Minimum Distance

To find the actual distance, we substitute $t = \frac{9}{25}$ back into the distance squared equation.

$$d^2 = 625 \left(\frac{9}{25} \right)^2 - 450 \left(\frac{9}{25} \right) + 225$$

$$d^2 = 625 \left(\frac{81}{625} \right) - 18(9) + 225$$

$$d^2 = 81 - 162 + 225$$

$$d^2 = 144$$

$$d = \sqrt{144} = 12 \text{ km}$$

⁴Video frame time interval: 01:55–02:01.

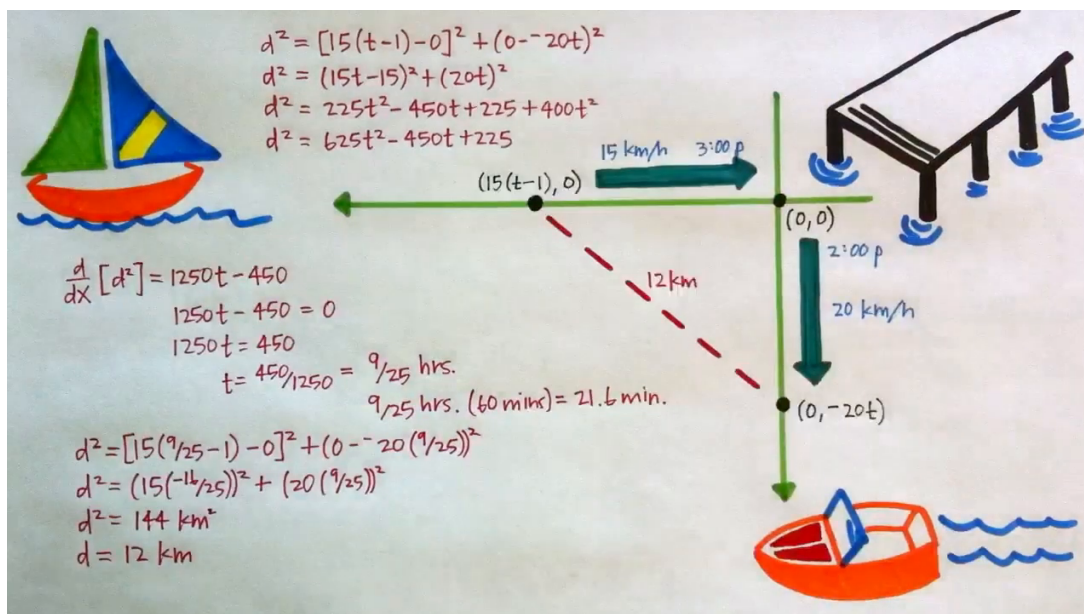


Figure 5: Final calculation of the minimum distance. ⁵

Final Result

The boats are closest together at **2:21:36 p.m.**, at which point the distance between them is **12 km**.

5 Summary and Extension

5.1 Summary

This problem demonstrates a standard approach to optimization in calculus:

1. **Model the situation:** Define a coordinate system and express the positions of moving objects as functions of time.
2. **Formulate the objective function:** Create an equation for the quantity to be optimized (in this case, distance squared).
3. **Differentiate and solve:** Find the critical points by setting the derivative to zero.
4. **Interpret the result:** Convert the mathematical solution back into the context of the problem (time and distance).

5.2 Extension

This type of problem is a specific case of "related rates" or "optimization of distance". Similar problems might involve:

- Objects moving at angles other than 90 degrees (requiring the Law of Cosines).
- Objects accelerating (requiring quadratic or higher-order position functions).
- Minimizing time or cost instead of distance.

For further practice with calculus problems, you can visit integralCALC.com.

⁵Video frame time interval: 02:11–02:15.